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SPECTRUM FILTERING BY MEANS OF NUMERICAL TRANSFORM THEOREM

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SPECTRUM FILTERING BY MEANS OF
NUMERICAL TRANSFORM THEOREM

by

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1 March 1966

CONTENTS

ILLUSTRATIONS

ABSTRACT

A new type of digital filtering is explained without using the convolution theorem used in electrical engineering. The scheme makes use of the Numerical Transform Theorem which transforms a pair of finite spectra (i.e., amplitude and phase spectra) in the frequency domain into a finite time series. This time series, which may be filtered during transformation, has precisely the same record length as the "original" time series without introducing meaningless transients. The spectrum filtering technique, in contrast to the convolution filtering technique, has an additional advantage that it does not require a separate digital filter. Some examples are given.

SPECTRUM FILTERING BY MEANS OF
NUMERICAL TRANSFORM THEOREM

ı. INTRODUCTION

For many years, Fourier series and transforms have been used in science and engineering. The primary reason lies in its simplicity of representations which allow us to analyze a function in a different form.

For purposes of numerical analyses, series truncation becomes inevitable, and expressions for a set of coefficients are found. There is an important theorem, however, which has escaped the attention of previous investigators. This theorem, which may be called "Numerical Transform Theorem", transforms finite time series to finite frequency harmonics.

The history of frequency filtering is probably as old as the history of electronic engineering. So far, digital filtering is accomplished through convolution theorem which states that a multiplication in a frequency domain corresponds to a convolution in the time domain. Theoretically, the convolution filter has infinite length, and a truncation of this filter at a certain tolerable level is necessitated in actual applications. The resulting filtered record contains useless tails caused by transients in the convolution process.

This article intends to show a different way of spectrum filtering by which a time series can be filtered without generating convolution transients. A terminology "Spectrum Filtering" is adopted to differentiate this technique from the conventional convolution filtering technique.

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2. NUMERICAL TRANSFORM THEOREM

Since the time Fourier made use of the trigonometric series bearing his name for the integration of the equations of thermal conduction, the technique has become one of the most basic tools in scientific and engineering analyses. The method involves an expression of a continuous function of length 2T by an infinite series,

$$f(t) = \frac{a'_0}{2} + \sum_{n=1}^{\infty} \left[a'_n \cos \frac{n\pi t}{T} + b'_n \sin \frac{n\pi t}{T} \right] \qquad (1)$$

where coefficients a'_n and b'_n can be determined by

$$\cdot'_n = \frac{1}{T} \int_0^{2T} f(t) \cos \frac{n\pi t}{T}\, dt \quad \text{and} \quad b'_n = \frac{1}{T} \int_0^{2T} f(t) \sin \frac{n\pi t}{T}\, dt \qquad (2)$$

Although these expressions are exact by virtue of the orthogonality relationship of trigonometric functions, it takes an infinite number of a'_n and b'_n to have an exact expression of the original f(t).

-1-

There is an important theorem in the theory of transforms, which is usually referred to as Fourier transform theorem (Lee, 1961, p. 33). The mathematical expression of this theorem is given by

$$f(t) = \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} f(\xi) e^{-j2\pi f(\xi-t)} d\xi \tag{3}$$

where the frequency is given in cycles per second.

This expression holds for continuous functions extending from $-\infty$ to ∞ in both time and frequency domains. The _approximation_ of the above expression in numerical form takes infinite terms, and it is difficult to recover the original function by using the equation (3).

In my previous presentation[1] at St. Louis, Missouri, I indicated a theorem called Numerical Transform Theorem (Huang, 1965). Borrowing the notation of Z-transform (Robinson and Treitel, 1964), this theorem can be expressed by

$$f_i = \frac{1}{m} \sum_{n=0}^{m-1} \sum_{k=0}^{m-1} f_k z^{n(k-i)} \tag{4}$$

where

$$z = e^{-j\frac{2\pi}{m}} \tag{5}$$

In the above expressions, j is an imaginary unit and m is the sample number. The expression (4) is also an exact relationship which transforms a _finite_ number of digitized data into a _finite_ domain. It indicates that the _exact_ recovery of the original digitized data is possible without consuming an infinite number of coefficients.

3. SPECTRUM FILTERING

By decomposing equation (3) into a transform pair, we have

$$F(f) = \int_{-\infty}^{\infty} f(\xi) e^{-j2\pi\xi f} d\xi \tag{6}$$

and

$$f(t) = \int_{-\infty}^{\infty} F(f) e^{j2\pi ft} df \tag{7}$$

Denoting a filter function by $H(f)$, the impulse response of this function is given by

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} df \tag{8}$$

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[1]Presented at the Seismological Society of America meeting, St. Louis, Missouri, on 14 April 1965.

TR 66-19

Since a frequency filtering is achieved through multiplication of F(f) and H(f) in the frequency domain, by the convolution theorem (Lee, 1961, p. 46), we obtain

$$\int_{-\infty}^{\infty} f(t - \tau)\, h\,(\tau)\, d\tau = \int_{-\infty}^{\infty} F(f)\, H(f) e^{j2\pi ft}\, df \qquad (9)$$

For digital data, this operation is accomplished through

$$g_i = \sum_k f_{i-k}\, h_k \qquad (10)$$

The h_k in the above expression represents a digital filter. For a filter length of $p\,\Delta t$ and a record length of $m\,\Delta t$, the resulting filtered record length is given by $(p + m - 1)\,\Delta t$, where Δt is the digitization increment. In this process, truncation of the $h(t)$ and a creation of the convolution transients are unavoidable.

Returning to the equation (4), the numerical transform theorem can be decomposed into

$$F_n = \frac{1}{m} \sum_{k=0}^{m-1} f_k\, z^{nk} \qquad (11)$$

and $\qquad f_i = \sum_{n=0}^{m-1} F_n\, z^{-ni} \qquad (12)$

Accordingly, when a spectrum filtering is performed through

$$F'_n = F_n\, H_n \qquad (13)$$

the filtered record f'_i can be represented by

$$f'_i = \sum_{n=0}^{m-1} F'_n\, z^{-ni} \qquad (14)$$

In equation (13), H_n represents a filtering function.

4. EXAMPLES OF SPECTRUM FILTERING

As an actual demonstration of the difference between the conventional and the new filtering techniques, the Apache deep-well record of August 3, 1965, is considered below. This record which is shown in figure 1 was taken at a depth of 9440 ft below the ground level. The record was digitized at a rate of 10 samples per second, and a record length of 15 seconds was used. Before digitization, the record was filtered through a band-pass filter of 0.5~5.0 cps with cut-off rates of 12 dB and 24 dB per octave, respectively.
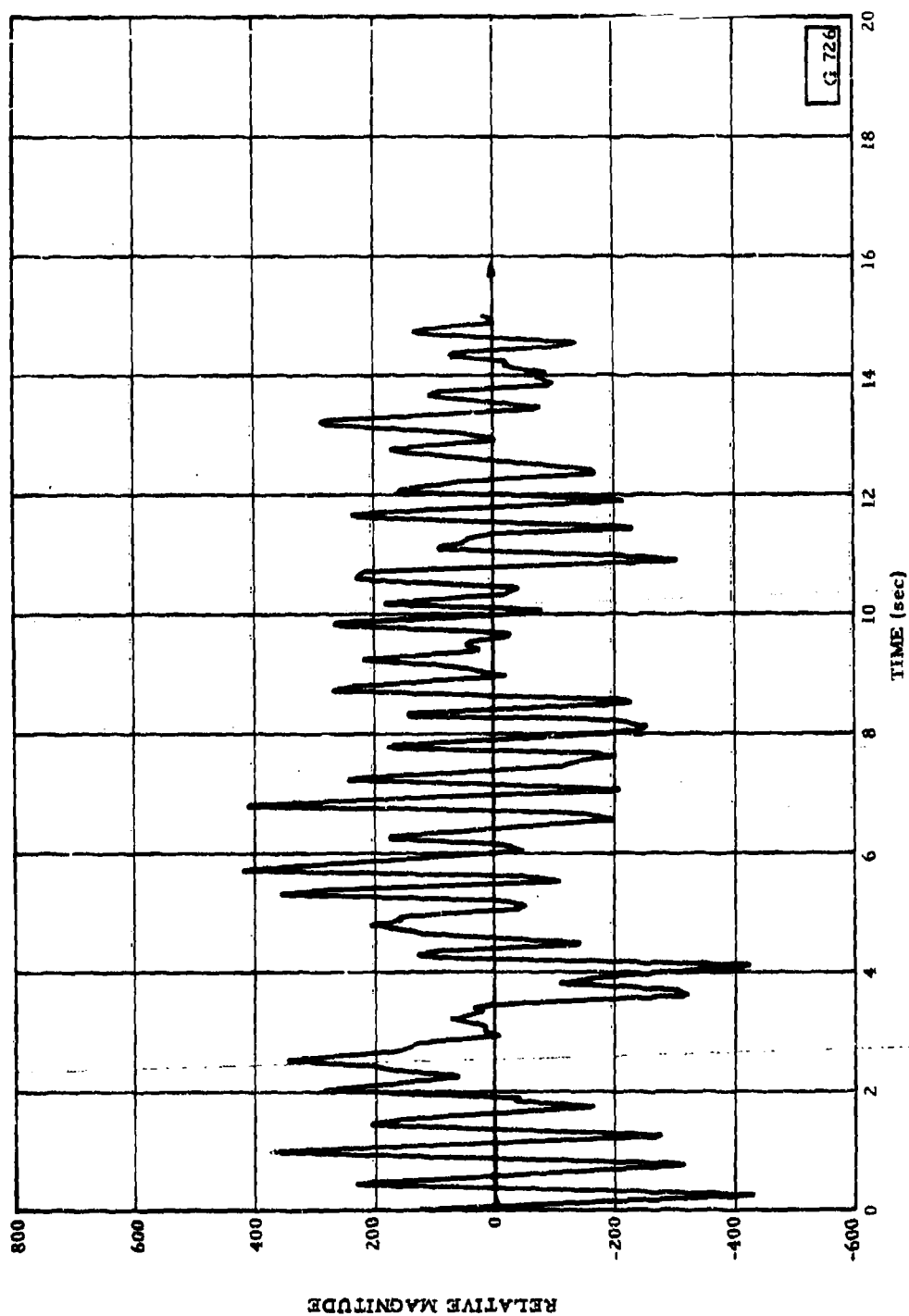
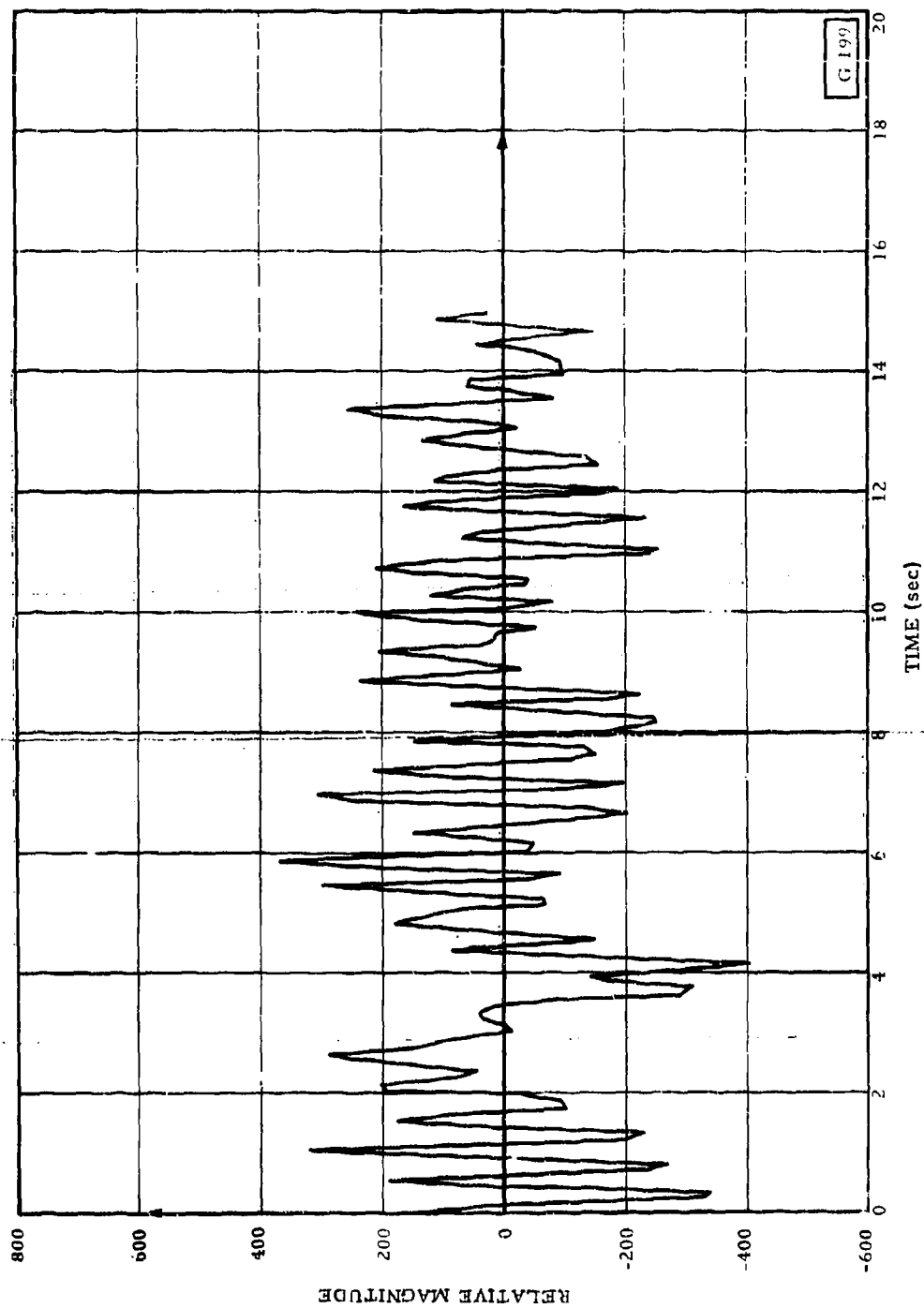Figure 1a. Analog deep-well record of August 3, 1965, at Apache, Oklahoma

Figure 1b. Digitized deep-well record of August 3, 1965 at Apache, Oklahoma

TR 66-19

2. Low-pass filter

For a filter length of $p\Delta t$, the fundamental frequency is given by

$$\Delta f = \frac{1}{p\Delta t} \qquad (15)$$

Any frequency f in terms of Δf can be expressed by

$$f_q = q\Delta f = \frac{q}{p\Delta t} \qquad (16')$$

The mathematical representation of a low-pass filter is

$$H_n = \begin{cases} 1, & \text{for } n = 0,1,2,\ldots N(N \leq q) \\ 0, & \text{for } n > N \end{cases} \qquad (17)$$

The impulse response of this filter is

$$h_i = \sum_{n=0}^{N} z^{-ni} \qquad (18)$$

Figure 2 shows the analog version of the low-pass filtered record (24 dB/octave at 1.5 cps) of the original figure 1a. An impulse response of the low-pass filter (cut-off at 1.5 cps) is shown in figure 3a and the digitally filtered result using the conventional convolution technique is shown in figure 3b.

Figure 4 is the corresponding low-pass filtered record by a direct usage of the numerical transform theorem.

b. High-pass filter

For a high-pass filter, the filtering function becomes

$$H_n = \begin{cases} 0, & \text{for } n = 0,1,2,\ldots N(N \leq q) \\ 1, & \text{for } n > N \end{cases} \qquad (19)$$

The impulse response of this filter is

$$h_i = \sum_{n>N}^{p-1} z^{-ni} \qquad (20)$$

Figure 5 is the analog high-pass filtered result. The impulse response of the high-pass filter together with the convolved result is shown in figures 6a and 6b. Figure 7 is the filtered output by using the numerical transform theorem.

c. Band-pass filter

The mathematical expression for a band-pass filter is given by

$$H_n = \begin{cases} 0, & \text{for } n = 0,1,2,\ldots N_1 \\ 1, & \text{for } n = N_1 + 1, N_1 + 2,\ldots N_2 \\ 0, & \text{for } n > N_2 \end{cases} \qquad (21)$$
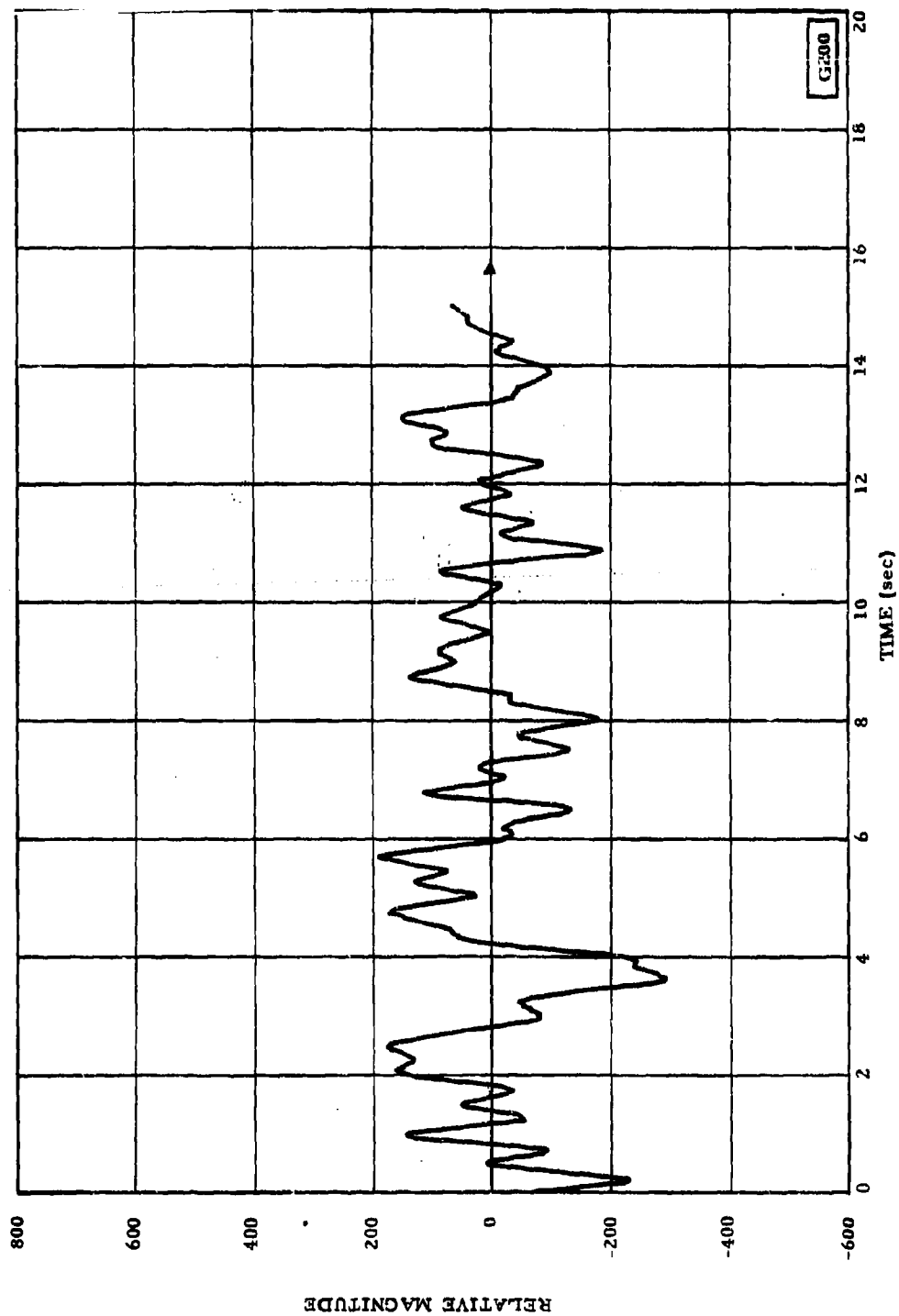
-6-

Figure 2. Analog low-pass filtered deep-well record, August 3, 1965
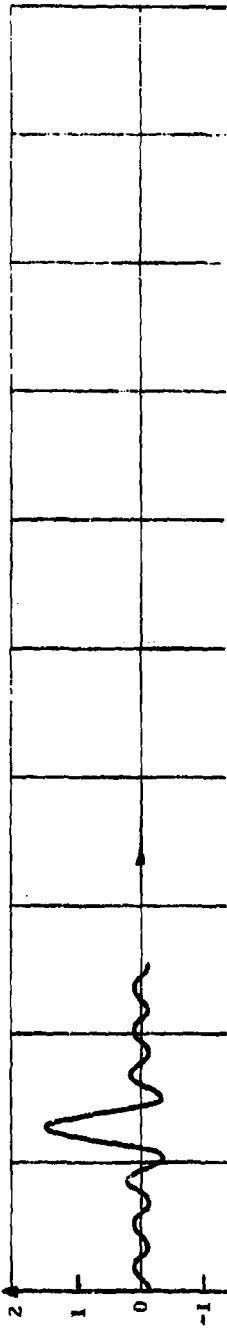
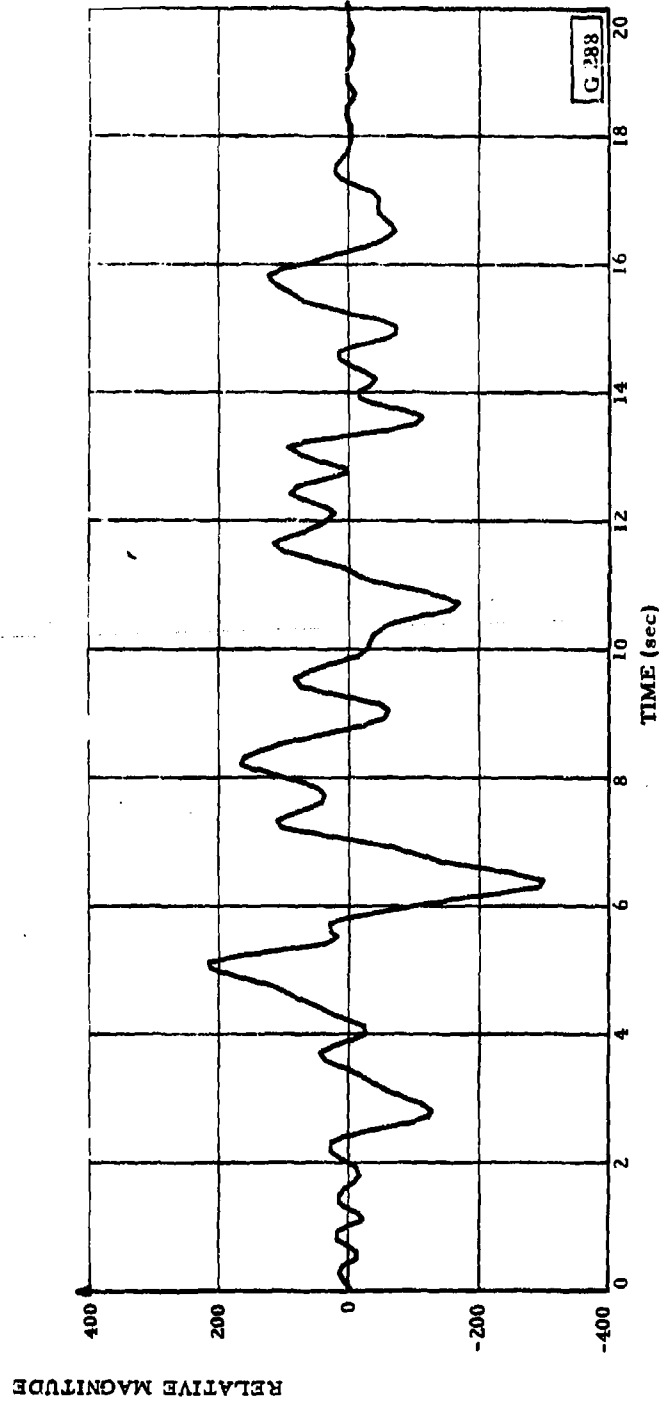Figure 3a. Impulse response of a low-pass filter (cut-off at 1.5 cps)

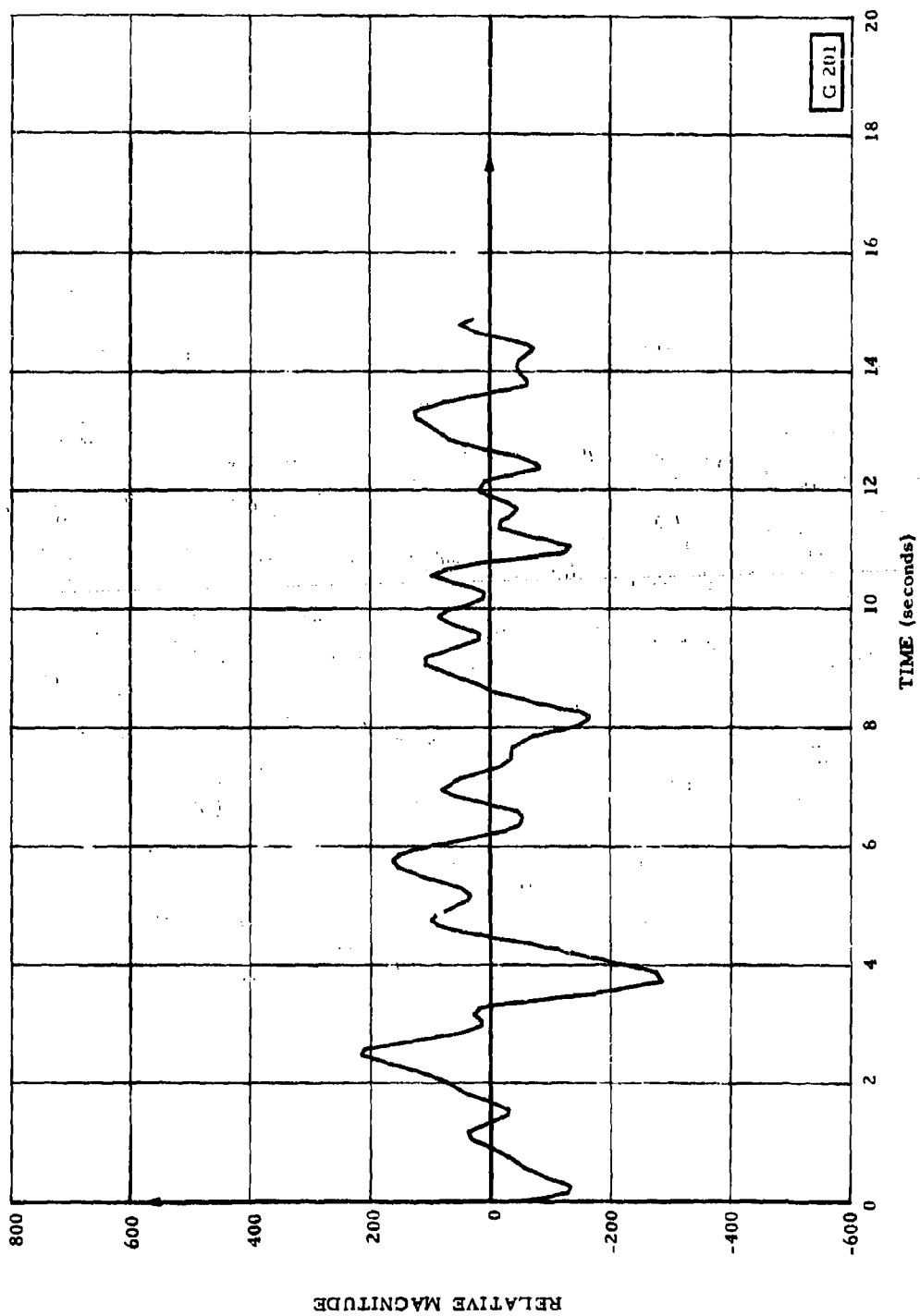Figure 3b. Low-pass filtered Apache record, using convolution theorem

-8-

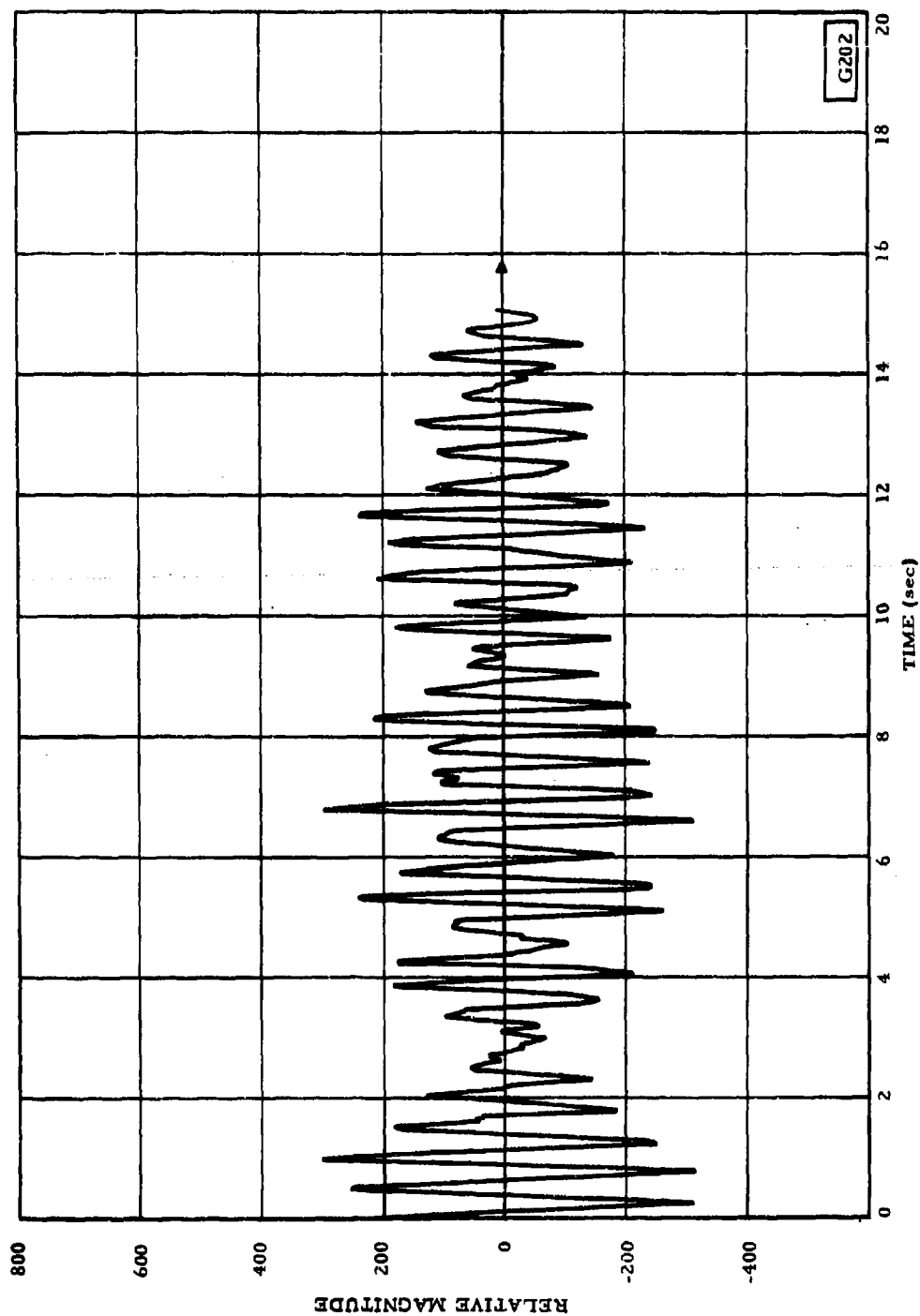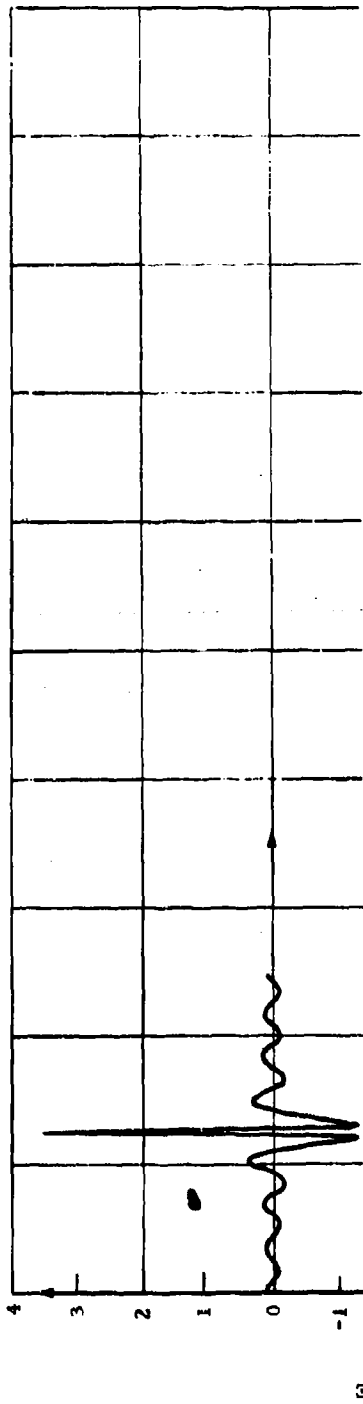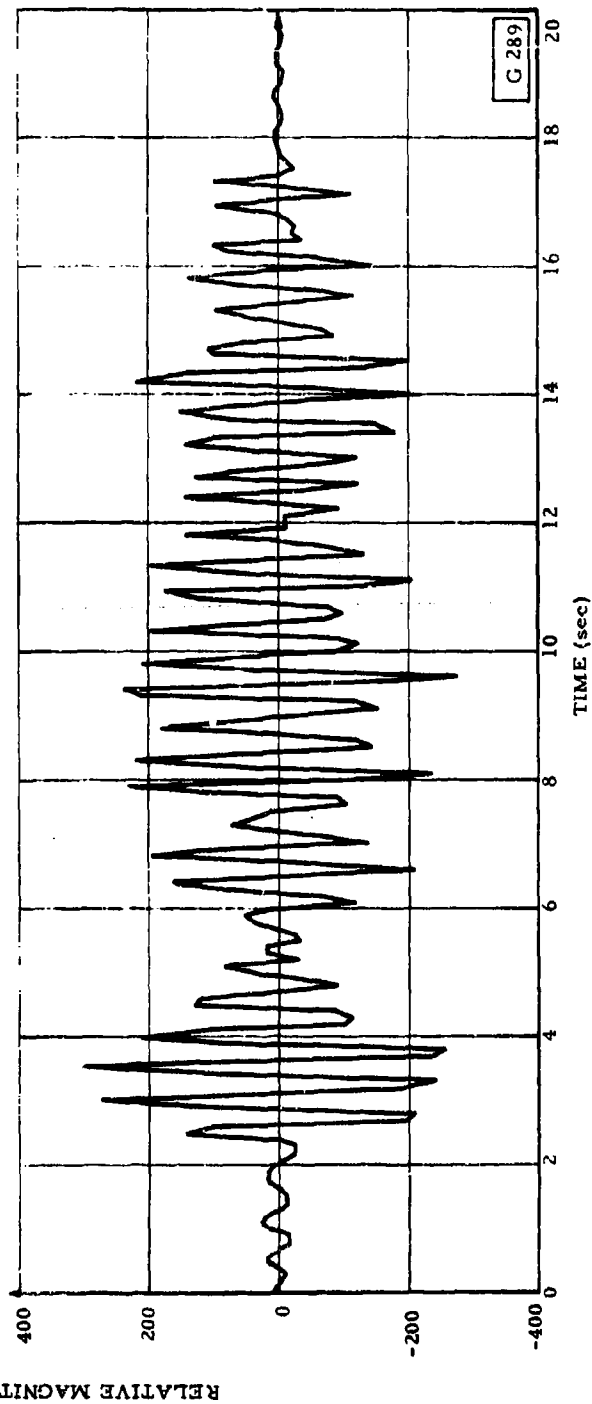Figure 4. Low-pass filtered Apache record using the numerical transform theorem

RELATIVE MAGNITUDE

TIME (seconds)

G 201

Figure 5. Analog high-pass filtered deep-well record, August 3, 1965

TR 66-19

Figure 6a. Impulse response of a high-pass filter (cut-off at 1.5 cps)

Figure 6b. High-pass filtered Apache record, using convolution theorem
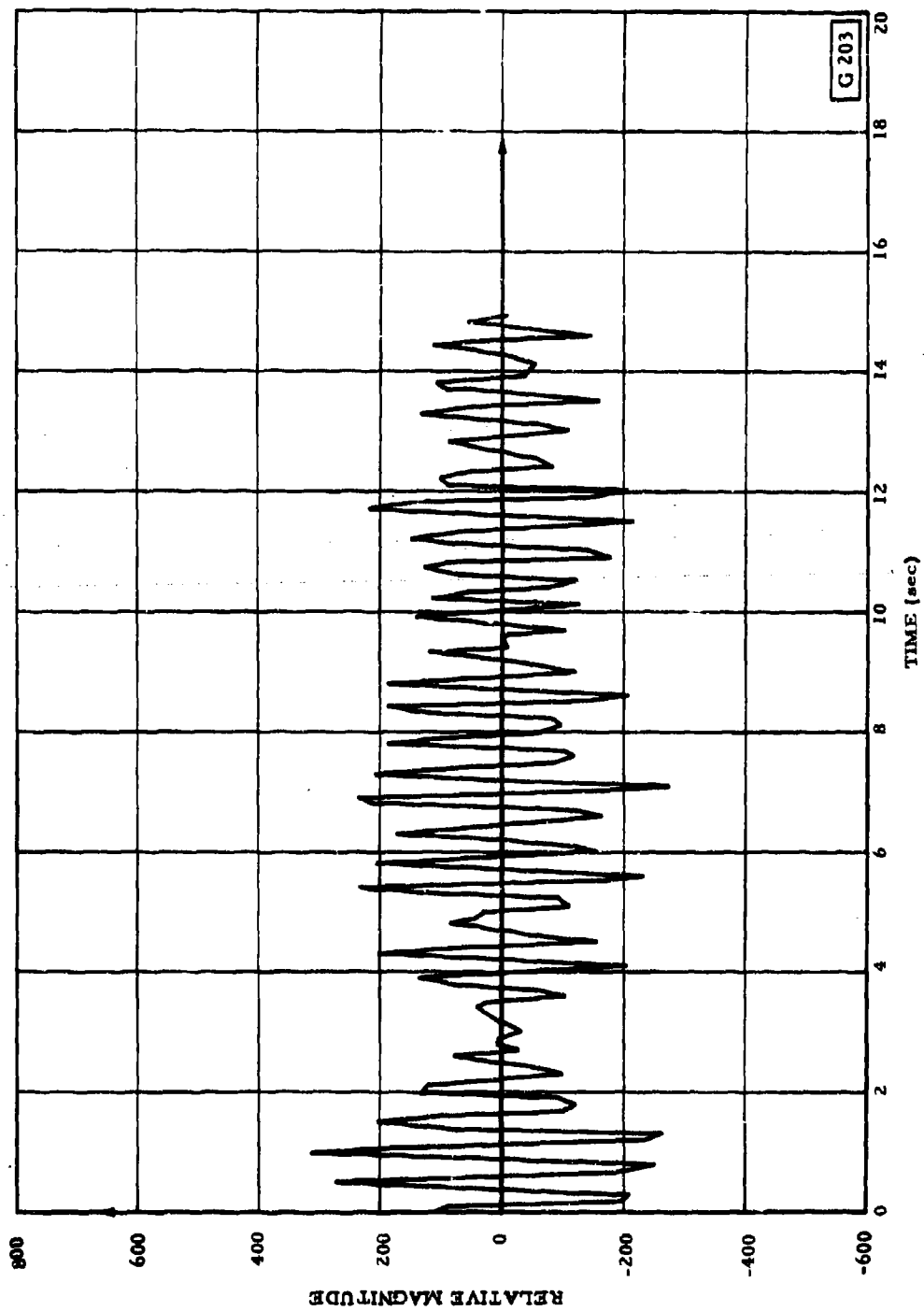
RELATIVE MAGNITUDE

TIME (sec)

G 289

-11-

TR 66-19

Figure 7. High-pass filtered Apache record using the numerical transform theorem

The impulse response of this filter is accordingly

$$h_i = \sum_{n > N_1}^{N_2} Z^{-ni}$$

(22)

Figures 8 through 10 show different results of the filtered outputs.

As a comparison between the analog results and the reconstructed results, figure 11 and figure 12 are displaye'. It is of interest to note that except for the dc bias, figure 12 is coincident with the figure 1b on which the spectrum filterings were performed. Other types of filtering, e.g., notch filter, comb filter, etc., can easily be arranged using the numerical transform theorem.

5. CONCLUSIONS

By virtue of the numerical transform theorem, it is possible to filter digitized records in any desired fashion in the frequency domain. No such useless transients are incurred in this process as in the case of filtering through convolutions.

When the filtered function using the numerical transform theorem is spectral analyzed using the Fourier transform, there will be small side lobes created between the multiples of the original fundamental frequency. This situation can be remedied by taking a sufficiently long record length initially. The increased frequency resolution will thus contribute to a reduction in the undesirable side lobes. This is the main trade-off between the analog and the digital filterings.

A comparison between figure 11 and figure 12 indicates that digital filtering is closer to a perfect filter than analog, and the application of the numerical transform theorem will yeild filtered records without transients. The transient effects are particularly significant for records of short durations.

Although one-dimensional transformation is shown in this article, the extention of this filtering technique can be accomplished to three-dimensional array problems where both time and space parameters must be taken into consideration. It must be remembered that the above technique requires spectral analysis before filtering, although a separate digital filter is not required in the process.

6. ACKNOWLEDGEMENT

7. REFERENCES

Lee, Y. W., "Statistical theory of communication", Wiley, p. 33, 1963.

Huang, Y. T., "Special analysis of digitized seismic data", Accepted for publication in the April, 1966, issue of the Bulletin of the Seismological Society of America.

Robinson, E. A., and Treitel, Sven, "Principles of digital filtering", Geophysics, p. 395-404, June, 1964.
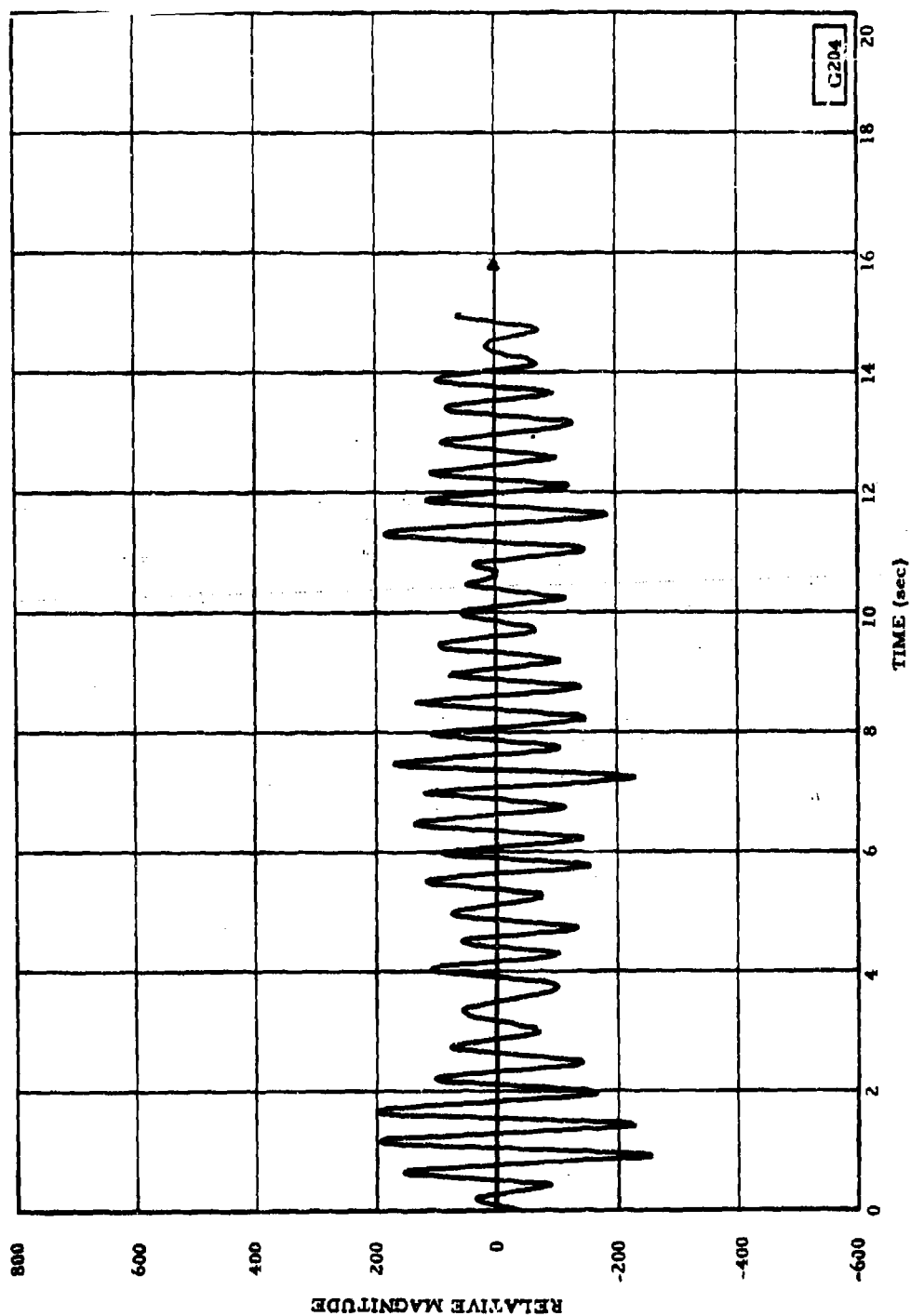
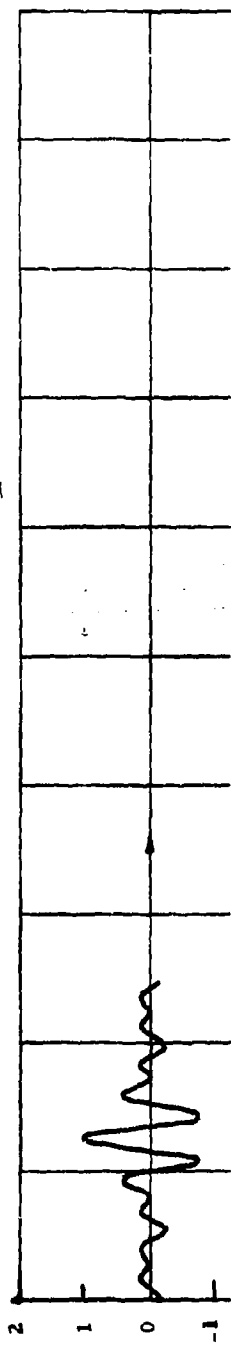Figure 8. Analog band-pass filtered deep-well record, August 3, 1965

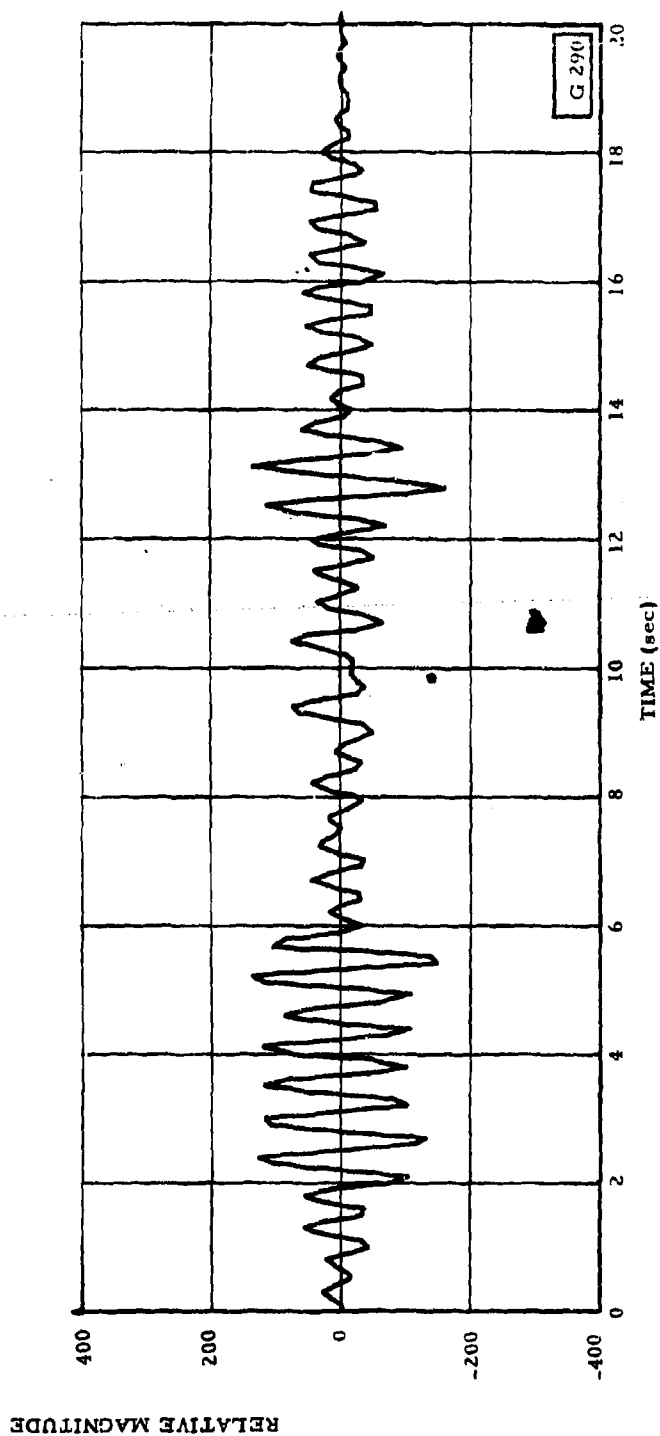Figure 9a. Impulse response of a band-pass filter (1 to 2 cps)

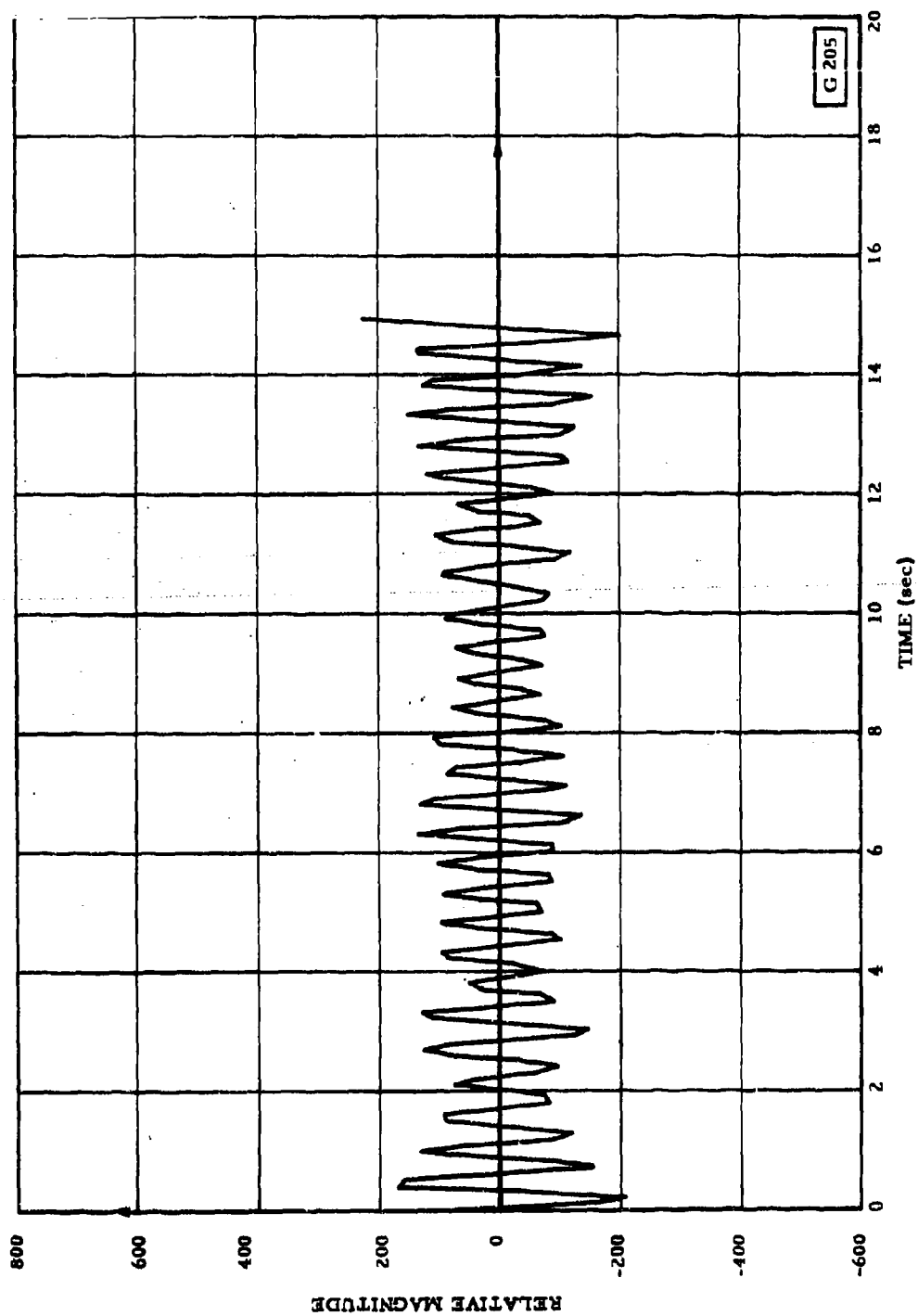Figure 9b. Band-pass filtered Apache record using convolution theorem

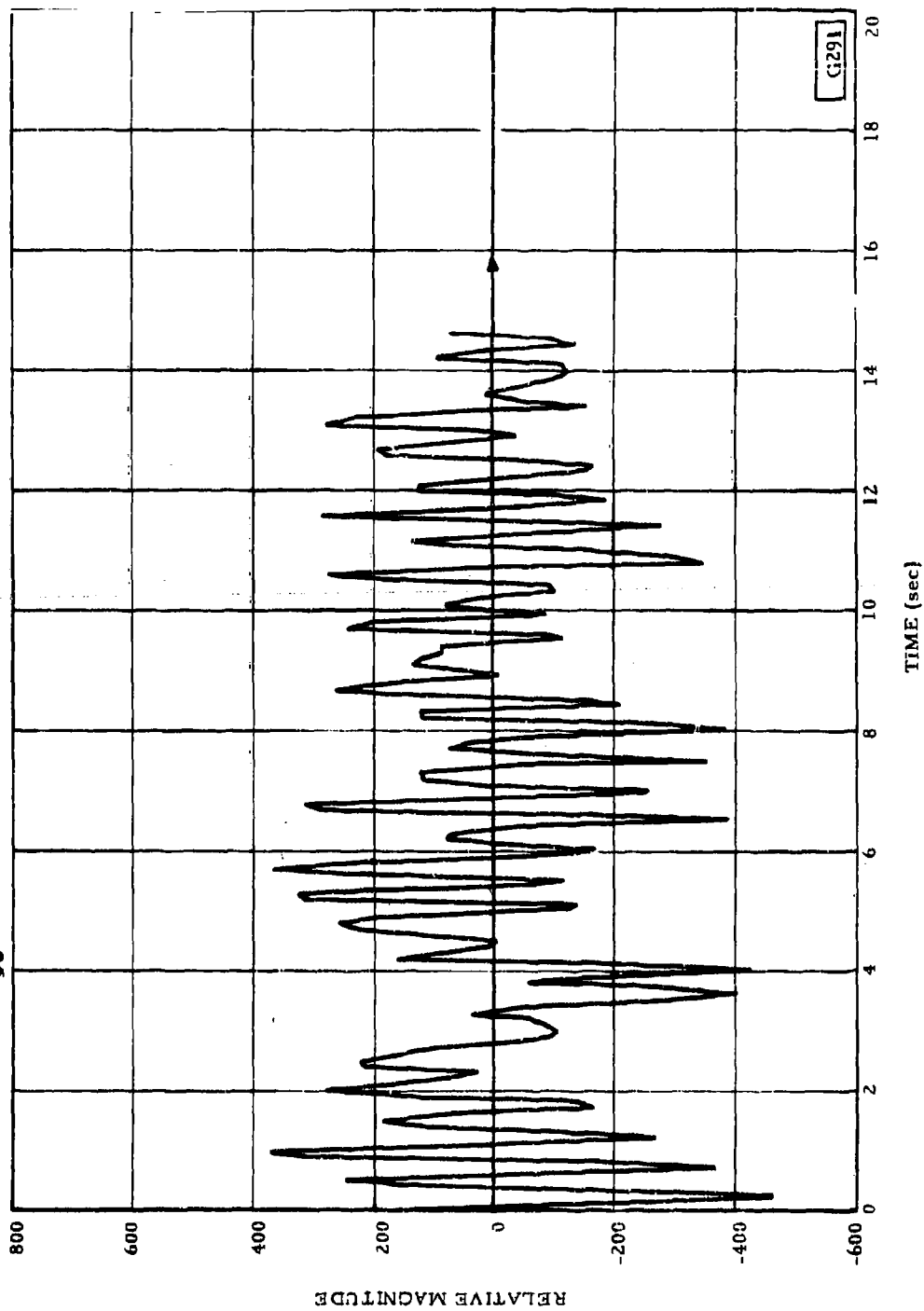Figure 10. Band-pass filtered analog record using the numerical transform theorem
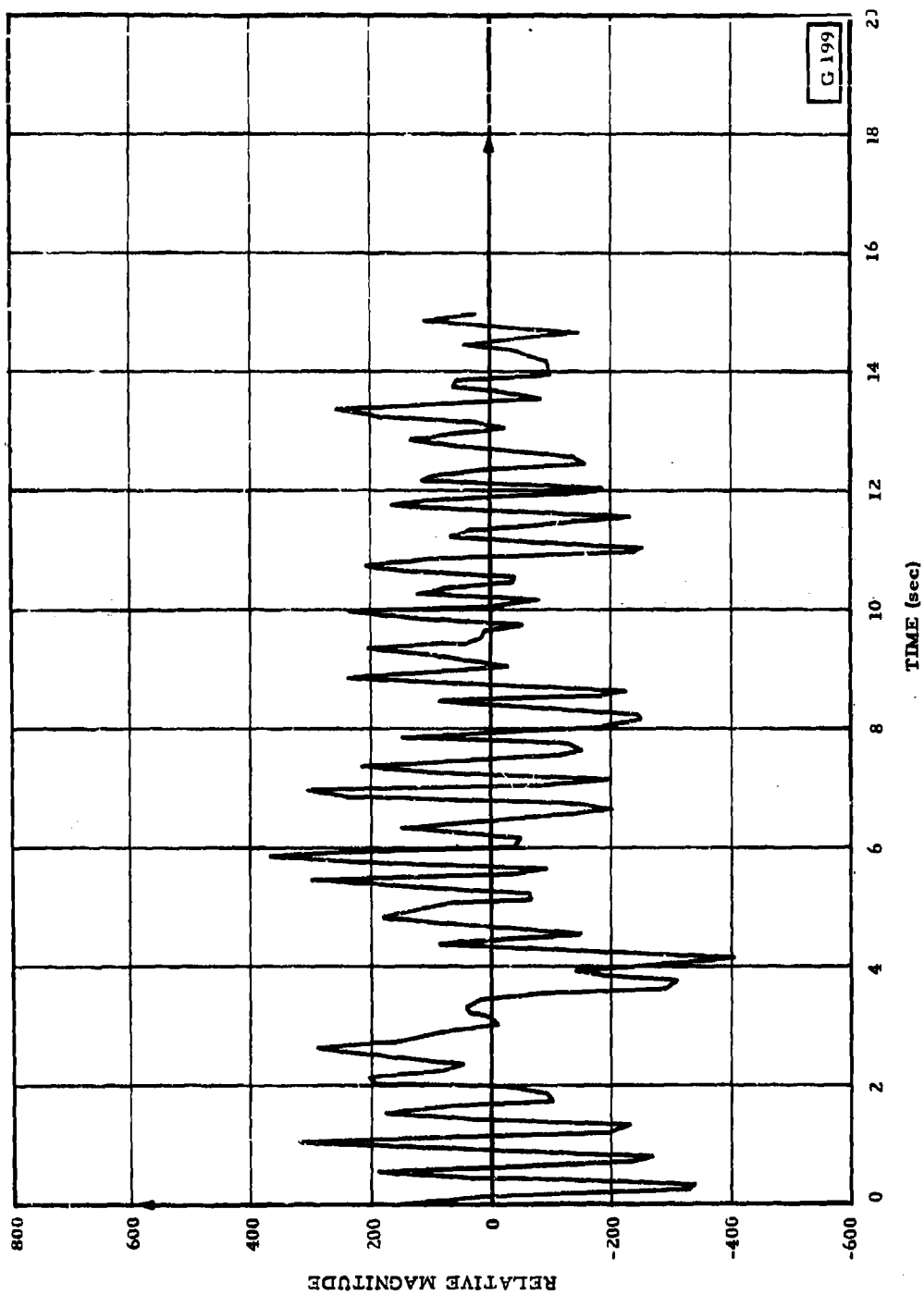
-16-

Figure 11. Addition of figure 2 and figure 5 after digitization

TR 66-19

Figure 12. Superposition of figure 4 and figure 7

TR 66-19